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VARIATIONAL PARAMETER ESTIMATION FOR A TWO-DIMENSIONAL NUMERICAL TIDAL MODEL

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SUMMARY

It is shown that the parameters in a two-dimensional (depth-averaged) numerical tidal model can be estimated accurately by assimilation of data from tide gauges. The tidal model considered is a semi-linearized one in which kinematical non-linearities are neglected but non-linear bottom friction is included. The parameters to be estimated (bottom friction coefficient and water depth) are assumed to be position-dependent and are approximated by piecewise linear interpolations between certain nodal values. The numerical scheme consists of a two-level leapfrog method. The adjoint scheme is constructed on the assumption that a certain norm of the difference between computed and observed elevations at the tide gauges should be minimized. It is shown that a satisfactory numerical minimization can be completed using either the Broyden–Fletcher–Goldfarb–Shanno (BFGS) quasi-Newton algorithm or Nash's truncated Newton algorithm. On the basis of a number of test problems, it is shown that very effective estimation of the nodal values of the parameters can be achieved provided the number of data stations is sufficiently large in relation to the number of nodes.

KEY WORDS Numerical tidal model Data assimilation Parameter estimation Optimal control

1. INTRODUCTION

The earliest numerical tidal models have been based on the vertically integrated continuity and momentum equations, and yield values of the surface elevation and depth-averaged velocity components. While these models have been, to some extent, superceded in the last fifteen years by full three-dimensional models, which are clearly superior for problems involving wind-driven and density-gradient-driven flows, they remain of considerable usefulness for tidal flows, for which the fluid velocity is not strongly dependent on the vertical co-ordinate, and for algorithms designed for small computers. Reviews of much of the work on numerical tidal models are given by Liu and Leendertse,[1] and Nihoul and Jamart.[2]

The parameters in the two-dimensional models are usually the water depth and the bottom friction coefficient, both of which are in general position-dependent. Traditionally, numerical tidal models are 'tuned' by adjusting these parameters so as to make the predicted surface elevations at certain tide stations agree as closely as possible with their observed values. In recent years systematic techniques of such data assimilation based on optimal control methods have been developed, particularly in the field of meteorology. These methods were originated by Sasaki[3,4] and Marchuk[5] and have more recently been developed and applied by Lewis and Derber,[6] Le Dimet and Talagrand,[7] Harlan and O'Brien,[8] Hoffman,[9] Lorenc,[10,11] Talagrand and Courtier[12] and Courtier and Talagrand.[13] Recent reviews of much of this work are given by Lorenc,[10] Navon[14] and Le Dimet and Navon.[15] Similar methods have also been used by

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Chavent *et al.*[16] and Carrera and Neumann[17] to estimate the parameters in models of flow in porous media.

In the field of oceanography such optimal control methods have also recently come into use. Bennett and McIntosh[18] and Prevost and Salmon[19] have applied the weak-constraint formalism of Sasaki[4] to a tidal flow problem and a geostrophic flow problem, respectively. More recently, the strong-constraint formalism has been used by Panchang and O'Brien[20] to determine the bottom friction coefficient in a problem of flow in a channel using some earlier experimental results. Smedstad[21] and Smedstad and O'Brien[22] have extended this approach and used it to determine the effective phase speed in a model of the equatorial Pacific Ocean based on observations of sea level. Yu and O'Brien[23] have used a similar method to estimate the eddy viscosity and surface-drag coefficient from the measured velocities of a wind-driven flow.

In a previous paper, Das and Lardner[24] have extended the work of Panchang and O'Brien to estimate the position-dependent drag and depth in a sectionally integrated model of flow in a channel by assimilation of periodic tidal data and have compared several minimization algorithms. Lardner[25] has used similar variational techniques to estimate the open boundary conditions in a two-dimensional tidal model. The aim of the present paper is to combine and extend the work in these two papers in order to develop the technique of parameter estimation in the two-dimensional case.

The variational method involves minimizing a certain functional which consists of a norm of the difference between the computed and measured values of surface elevation at the tide gauges within the water body. An algorithm is obtained, via the so-called adjoint equations, for construction of the gradient of this functional with respect to the parameters. Having determined the gradient, the minimization can be performed using any of a number of numerical optimization algorithms. Here we have used both the Broyden–Fletcher–Goldfarb–Shanno (BFGS) quasi-Newton algorithm as contained in the CONMIN subroutine of Shanno and Phua[26] and Nash's truncated Newton algorithm.[27] Descriptions of these two methods are given by Navon and Legler,[28] and Nash and Nocedal.[29]

In Section 2 we describe the numerical tidal model that we have used. In Section 3 the corresponding discrete adjoint is constructed and the parameter equations derived. In Section 4 the results of several numerical tests are given. Section 5 summarizes the results and conclusions.

2. THE NUMERICAL TIDAL MODEL

2.1. Basic equations

We let x, y be the horizontal co-ordinates and t be the time. The undisturbed depth of the water at position (x, y) is denoted by $h(x, y)$, the elevation of the free surface above its undisturbed position by $\zeta(x, y, t)$ and the depth-averaged components of water velocity in the x- and y-directions by $u(x, y, t)$ and $v(x, y, t)$.

As in most tidal models, we make the usual hydrostatic approximation, assume the fluid to be incompressible and of uniform density, and neglect horizontal shear stresses. We shall also neglect the advective terms in the momentum equations, which in most tidal flows are quite small. Rather than u and v, we shall use the components of volume transport, $p=(h+\zeta)u\approx hu$ and $q=(h+\zeta)v\approx hv$ as primary variables. Then the depth-integrated equations of continuity and momentum can be written as

$$\zeta_t + p_x + q_y = 0,$$

$$p_t - fq + gh\zeta_x + s^{(x)} = 0,$$

$$q_t + fp + gh\zeta_y + s^{(y)} = 0,$$

where f is the Coriolis parameter and $s^{(x)}$, $s^{(y)}$ are the components of bottom friction divided by water density. These latter are assumed to have the usual quadratic form,

$$(s^{(x)}, s^{(y)}) = kh^2\sqrt{(u^2+v^2)}(u, v) = k\sqrt{(p^2+q^2)}(p, q),$$

where $k(x, y)$ is a bottom friction coefficient. The factor h^2 is included for convenience. The equations can then be re-expressed as follows:

$$\zeta_t + p_x + q_y = 0, \tag{1}$$

$$p_t - fq + gh\zeta_x + kp\sqrt{(p^2+q^2)} = 0, \tag{2}$$

$$q_t + fp + gh\zeta_y + kq\sqrt{(p^2+q^2)} = 0. \tag{3}$$

The boundary conditions on a coastal boundary are taken to be such that the normal component of vector (p, q) is zero. On an open boundary, we assume that the surface elevation ζ is known. With these boundary conditions plus the initial values of ζ, p and q, equations (1)–(3) form a well-posed boundary value problem.

2.2. *Finite difference approximations*

The numerical scheme is based on a leapfrog method with staggered grids in both space and time. The spatial grid is identical with an Arakawa C-grid, but the variables ζ and (p, q) are taken at alternating half-steps in the time direction. At time step j, the value of ζ at grid point (m, n) is denoted by $\zeta^j_{m,n}$, while at time step $j+\frac{1}{2}$, the value of p at grid point $(m+\frac{1}{2}, n)$ is denoted by $p^j_{m,n}$ and the value of q at grid point $(m, n+\frac{1}{2})$ is denoted by $q^j_{m,n}$. This scheme has previously been used for the tidal equations by Lardner and Smoczynski.[30]

The finite difference approximation to equation (1), centred at the space–time grid point $(m, n, j+\frac{1}{2})$, is then

$$\frac{\zeta^{j+1}_{m,n} - \zeta^j_{m,n}}{\Delta t} + \frac{p^j_{m,n} - p^j_{m-1,n}}{\Delta x} + \frac{q^j_{m,n} - q^j_{m,n-1}}{\Delta y} = 0, \tag{4}$$

where Δx, Δy and Δt are the spatial and temporal grid spacings. Since all differences are centred, the local truncation error in this approximation is second order. Equation (4) may be solved explicitly for $\zeta^{j+1}_{m,n}$

The finite difference approximations to equations (2) and (3) are as follows:

Equation (2) *centred at the space–time point* $(m+\frac{1}{2}, n, j+1)$.

$$\frac{p^{j+1}_{m,n} - p^j_{m,n}}{\Delta t} - f\bar{q}^{[j]}_{m,n} + gh_{m+1/2,n}\frac{(\zeta^{j+1}_{m+1,n} - \zeta^{j+1}_{m,n})}{\Delta x} + k_{m+1/2,n}r^j_{m,n}\{\alpha p^{j+1}_{m,n} + (1-\alpha)p^j_{m,n}\} = 0. \tag{5}$$

Equation (3) *centred at the space–time point* $(m, n+\frac{1}{2}, j+1)$:

$$\frac{q^{j+1}_{m,n} - q^j_{m,n}}{\Delta t} - f\bar{p}^{(j)}_{m,n} + gh_{m,n+1/2}\frac{(\zeta^{j+1}_{m,n+1} - \zeta^{j+1}_{m,n})}{\Delta y} + k_{m,n+1/2}s^j_{m,n}\{\alpha q^{j+1}_{m,n} + (1-\alpha)q^j_{m,n}\} = 0. \tag{6}$$

Equations (5) and (6) may be solved explicitly for $p^{j+1}_{m,n}$ and $q^{j+1}_{m,n}$, respectively. Here we use the notation

$$r^j_{m,n} = \sqrt{[(p^j_{m,n})^2 + (\bar{q}^j_{m,n})^2]}, \qquad s^j_{m,n} = \sqrt{[(\bar{p}^j_{m,n})^2 + (q^j_{m,n})^2]}. \tag{7}$$

In these expressions as well as in the Coriolis terms we use the usual four-point averages,

$$\bar{p}^j_{m,n} = \tfrac{1}{4}(p^j_{m,n} + p^j_{m-1,n} + p^j_{m,n+1} + p^j_{m-1,n+1}),$$

$$\bar{q}^j_{m,n} = \tfrac{1}{4}(q^j_{m,n} + q^j_{m+1,n} + q^j_{m,n-1} + q^j_{m+1,n-1}), \tag{8}$$

In addition, the notation

$$p_{m,n}^{[j]}=\begin{cases}p_{m,n}^{j} & \text{if } j \text{ is odd,}\\ p_{m,n}^{j+1} & \text{if } j \text{ is even,}\end{cases}\qquad q_{m,n}^{[j]}=\begin{cases}q_{m,n}^{j+1} & \text{if } j \text{ is odd,}\\ q_{m,n}^{j} & \text{if } j \text{ is even.}\end{cases}\tag{9}$$

is used. This denotes the fact that on even time steps equation (5) is solved first for $p_{m,n}^{j+1}$, while on odd steps equation (6) is solved first for $q_{m,n}^{j+1}$, and in each case the latest available values of the other variable are used in the Coriolis terms. It can be shown that this treatment of the Coriolis terms has second-order truncation error and also places an insignificant stability restriction on the time step.

In the discretizations (5) and (6), the bottom friction terms are treated semi-implicitly in order to improve stability. The implicitness parameter is denoted by α and, in practice, we have taken $\alpha=\frac{1}{2}$, which gives a centred average for this factor.

A small but significant change must be made in the appropriate one of the averages (8) when the velocity point (m, n) is adjacent to an open boundary. A typical situation is illustrated in Figure 1. In forming $\bar{q}_{m,n}$, $q_{m+1,n}$ and $q_{m+1,n-1}$ are not available, so only a one-sided average can be used: $\bar{q}_{m,n}^{j}=\frac{1}{2}(q_{m,n}^{j}+q_{m,n-1}^{j})$. Similar one-sided averages occur adjacent to any open boundary.

Boundary conditions are imposed on the discrete equations in the usual way for an Arakawa C-grid (see, for example, Reference 31). The grid is positioned so that coastal boundaries that run in the y-direction pass through p points with the values of p at such points being set equal to zero,

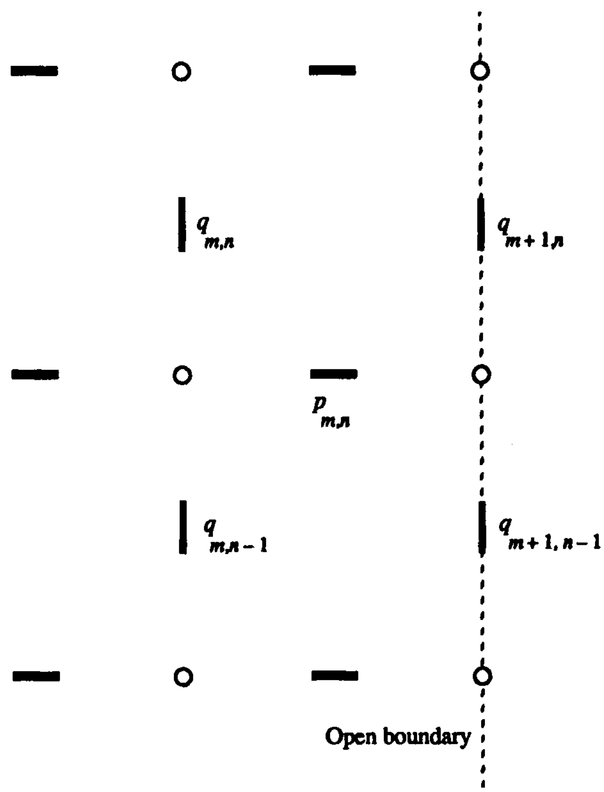

Figure 1. Example of a p point adjacent to a right-hand open boundary. For $\bar{q}_{m,n}$ only a two-point average of $q_{m,n}$ and $q_{m,n-1}$ is used

and coastal boundaries that run in the x-direction pass through q points with the corresponding values of q being set equal to zero.

Open boundaries are positioned so as to pass through ζ points, at which the values of ζ are set equal to the prescribed elevations. Let the open boundary points be denoted by (m_l, n_l), where $l = 1, 2, \ldots, L$. In general, at time step j the open boundary condition is taken as

$$\zeta^j_{m_l, n_l} = a_{0,l} + \sum_{i=1}^{M} a_{i,l} \cos(\omega_i j \Delta t - \phi_{i,l}), \tag{10}$$

where $\{\omega_i : i = 1, \ldots, M\}$ are the angular frequencies of the tidal constituents that are included in the model and $\{a_{i,l}, \phi_{i,l} : i = l, \ldots, M\}$ the amplitudes and phases of these constituents at the boundary point l.

Finally, as is usual with numerical tidal models, flat initial conditions are used, that is,

$$\zeta^0_{m,n} = p^0_{m,n} = q^0_{m,n} = 0.$$

3. THE ADJOINT NUMERICAL MODEL

3.1. Minimum principle

We suppose that the values of surface elevation Z^j_d are observed at certain tide stations labelled by $d = 1, 2, \ldots, D$ and for time steps $j = I + 1, \ldots, J$. The start-up interval consisting of steps $j = 1, 2, \ldots, I$ is imposed to allow the transients arising from the initial conditions to become sufficiently small, so that the computed solution should agree with the observed values to within some tolerance in the window $I + 1 \leq j \leq J$.

The tide stations may coincide with grid points but, more generally, we suppose that the corresponding computed values of elevation at station d are given by $\sum_{m,n} B_{m,n,d} \zeta^j_{m,n}$, where $B_{m,n,d}$ are appropriate interpolation coefficients. The discrepancy in the computed value at station d and step j is then

$$\Delta^j_d \equiv \sum_{m,n} B_{m,n,d} \zeta^j_{m,n} - Z^j_d. \tag{11}$$

We suppose that the parameters in the model must be chosen so as to minimize the objective function

$$F = \frac{1}{2} \sum_{d=1}^{D} K_d \sum_{j=I+1}^{J} (\Delta^j_d)^2, \tag{12}$$

where the quantities K_d are the respective weights given to the observational discrepancies at the different data stations.

Introducing Lagrange multipliers, $\lambda^j_{m,n}$, $\mu^j_{m,n}$ and $\nu^j_{m,n}$ for the constraints (4)–(6), we have the following expression for the first variation of F:

$$\begin{aligned}
\delta F = & \sum_{d=1}^{D} K_d \sum_{j=I+1}^{J} \Delta^j_d \sum_{m,n} B_{m,n,d}\, \delta\zeta^j_{m,n} \\
& + \sum_{j=0}^{J} \sum_{(m,n)\in S_\zeta} \lambda^j_{m,n}\, \delta \left\{ \frac{\zeta^{j+1}_{m,n} - \zeta^j_{m,n}}{\Delta t} + \frac{p^j_{m,n} - p^j_{m-1,n}}{\Delta x} + \frac{q^j_{m,n} - q^j_{m,n-1}}{\Delta y} \right\} \\
& + \sum_{j=0}^{J} \sum_{(m,n)\in S_p} \mu^{j+1}_{m,n}\, \delta \left\{ \frac{p^{j+1}_{m,n} - p^j_{m,n}}{\Delta t} - f\bar{q}^{[j]}_{m,n} + k_{m+1/2,n} r^j_{m,n} \{\alpha p^{j+1}_{m,n} + (1-\alpha) p^j_{m,n}\} \right.
\end{aligned}$$

$$+gh_{m+1/2,n}\frac{(\zeta^{j+1}_{m+1,n}-\zeta^{j+1}_{m,n})}{\Delta x}\Bigg\}$$

$$+\sum_{j=0}^{J}\sum_{(m,n)\in S_q} v^{j+1}_{m,n}\,\delta\Bigg\{\frac{q^{j+1}_{m,n}-q^{j}_{m,n}}{\Delta t}-f\bar{p}^{(j)}_{m,n}+k_{m,n+1/2}\,s^{j}_{m,n}\{\alpha q^{j+1}_{m,n}+(1-\alpha)q^{j}_{m,n}\}$$

$$+gh_{m,n+1/2}\frac{(\zeta^{j+1}_{m,n+1}-\zeta^{j+1}_{m,n})}{\Delta y}\Bigg\}. \tag{13}$$

Here S_ζ, S_p and S_q denote the sets of index pairs (m, n) at which equations (4), (5) and (6), respectively, are imposed. These sets do not include points on either closed or open boundaries.

Bearing in mind the grid points at which the difference approximations (4)–(6) are centred, we see that the multipliers λ, μ and v are discretized at the same spatial grid points as ζ, p and q, respectively.

After transforming certain of the summation indices in the last three terms of δF and taking account of the boundary and initial conditions, we can rewrite δF in the form

$$\delta F=\sum_{d=1}^{D}K_d\sum_{j=I+1}^{J}\Delta^j_d\sum_{m,n}B_{m,n,d}\,\delta\zeta^j_{m,n}$$

$$+\sum_{j=0}^{J}\sum_{(m,n)\in S_\zeta}W^j_{m,n}\,\delta\zeta^j_{m,n}+\sum_{j=0}^{J}\sum_{(m,n)\in S_p}U^j_{m,n}\,\delta p^j_{m,n}+\sum_{j=0}^{J}\sum_{(m,n)\in S_q}V^j_{m,n}\,\delta q^j_{m,n}$$

$$+\sum_{(m,n)\in S_p}\{P_{m,n}\,\delta k_{m+1/2,n}+R_{m,n}\,\delta h_{m+1/2,n}\}+\sum_{(m,n)\in S_q}\{Q_{m,n}\,\delta k_{m,n+1/2}+S_{m,n}\,\delta h_{m,n+1/2}\}, \tag{14}$$

where U, V and W are certain coefficients to be given below and

$$P_{m,n}=\sum_{j=0}^{J}\mu^{j+1}_{m,n}\,r^j_{m,n}\{\alpha p^{j+1}_{m,n}+(1-\alpha)p^j_{m,n}\},\quad (m,n)\in S_p,$$

$$Q_{m,n}=\sum_{j=0}^{J}v^{j+1}_{m,n}\,s^j_{m,n}\{\alpha q^{j+1}_{m,n}+(1-\alpha)q^j_{m,n}\},\quad (m,n)\in S_q, \tag{15}$$

$$R_{m,n}=\sum_{j=0}^{J}\mu^{j+1}_{m,n}\,\frac{g}{\Delta x}\{\zeta^{j+1}_{m+1,n}-\zeta^{j+1}_{m,n}\},\quad (m,n)\in S_p$$

$$S_{m,n}=\sum_{j=0}^{J}v^{j+1}_{m,n}\,\frac{g}{\Delta y}\{\zeta^{j+1}_{m,n+1}-\zeta^{j+1}_{m,n}\},\quad (m,n)\in S_q. \tag{16}$$

The quantities $r^j_{m,n}$ and $s^j_{m,n}$ are defined in equation (7). The details of this transformation are given by Lardner.[25] In order to reduce δF to this form, it is necessary to impose the final conditions that $\lambda^J_{m,n}=\mu^{J+1}_{m,n}=v^{J+1}_{m,n}=0$ and the boundary conditions that $\lambda^j_{m,n}=0$ at every open boundary point, $\mu^j_{m,n}=0$ at every coastal boundary that runs in the y-direction and $v^j_{m,n}=0$ at every coastal boundary running in the x-direction.

Setting the coefficients of the variations $\delta\zeta^j_{m,n}$, $\delta p^j_{m,n}$ and $\delta q^j_{m,n}$ equal to zero, we obtain the equations

$$W^j_{m,n}+\Theta^j\sum_{d=1}^{D}K_d\,B_{m,n,d}\,\Delta^j_d=0,\quad (m,n)\in S_\zeta,$$

$$U^j_{m,n}=0,\quad (m,n)\in S_p,$$

$$V^j_{m,n}=0,\quad (m,n)\in S_q,$$

where $\Theta^j=1$ if $I+1\le j\le J$, and $\Theta^j=0$ if $0\le j\le I$. These equations, together with the above boundary and final conditions, form the adjoint boundary value problem. In full they have the following forms:

$$\frac{\lambda^{j-1}_{m,n}-\lambda^j_{m,n}}{\Delta t}+\frac{g}{\Delta x}\{\mu^j_{m-1,n}h_{m-1/2,n}-\mu^j_{m,n}h_{m+1/2,n}\}$$
$$+\frac{g}{\Delta y}\{v^j_{m,n-1}h_{m,n-1/2}-v^j_{m,n}h_{m,n+1/2}\}+\Theta^j\sum_{d=1}^{D}K_dB_{m,n,d}\Delta^j_d=0, \tag{17}$$

$$\frac{\mu^j_{m,n}-\mu^{j+1}_{m,n}}{\Delta t}+\frac{\lambda^j_{m,n}-\lambda^j_{m+1,n}}{\Delta x}+\tilde{\eta}^{j+1}_{m,n}$$
$$+k_{m+1/2,n}\left\{\alpha\mu^j_{m,n}r^{j-1}_{m,n}+(1-\alpha)\mu^{j+1}_{m,n}r^j_{m,n}+\mu^{j+1}_{m,n}\frac{p^j_{m,n}[\alpha p^{j+1}_{m,n}+(1-\alpha)p^j_{m,n}]}{r^j_{m,n}}\right\}=0, \tag{18}$$

$$\frac{v^j_{m,n}-v^{j+1}_{m,n}}{\Delta t}+\frac{\lambda^j_{m,n}-\lambda^j_{m,n+1}}{\Delta y}+\tilde{\xi}^{j+1}_{m,n}$$
$$+k_{m,n+1/2}\left\{\alpha v^j_{m,n}s^{j-1}_{m,n}+(1-\alpha)v^{j+1}_{m,n}s^j_{m,n}+v^{j+1}_{m,n}\frac{q^j_{m,n}[\alpha q^{j+1}_{m,n}+(1-\alpha)q^j_{m,n}]}{s^j_{m,n}}\right\}=0. \tag{19}$$

Here

$$\xi^{j+1}_{m,n}=k_{m+1/2,n}\mu^{j+1}_{m,n}\frac{\bar{q}^j_{m,n}}{r^j_{m,n}}\{\alpha p^{j+1}_{m,n}+(1-\alpha)p^j_{m,n}\}-f\mu^{(j)}_{m,n},$$
$$\eta^{j+1}_{m,n}=k_{m,n+1/2}v^{j+1}_{m,n}\frac{\bar{p}^j_{m,n}}{s^j_{m,n}}\{\alpha q^{j+1}_{m,n}+(1-\alpha)q^j_{m,n}\}+fv^{\langle j\rangle}_{m,n}, \tag{20}$$

where

$$\mu^{(j)}_{m,n}=\begin{cases}\mu^j_{m,n}+\mu^{j+1}_{m,n} & \text{if } j \text{ is odd,}\\ 0 & \text{if } j \text{ is even,}\end{cases}\qquad v^{\langle j\rangle}_{m,n}=\begin{cases}0 & \text{if } j \text{ is odd,}\\ v^j_{m,n}+v^{j+1}_{m,n} & \text{if } j \text{ is even.}\end{cases} \tag{21}$$

The averages $\tilde{\xi}^j_{m,n}$ and $\tilde{\eta}^j_{m,n}$ are identical to the four point averages defined in equations (8) except at velocity points that are adjacently parallel to an open boundary. The four exceptional configurations are illustrated in Figure 2, and in these cases the new averages are defined as follows:

$$\begin{aligned}
&\text{Case A:}\quad \tilde{\mu}^j_{m,n}=\tfrac{1}{4}(\mu^j_{m,n}+2\mu^j_{m-1,n}+\mu^j_{m,n+1}+2\mu^j_{m-1,n+1});\\
&\text{Case B:}\quad \tilde{\mu}^j_{m,n}=\tfrac{1}{4}(2\mu^j_{m,n}+\mu^j_{m-1,n}+2\mu^j_{m,n+1}+\mu^j_{m-1,n+1});\\
&\text{Case C:}\quad \tilde{v}^j_{m,n}=\tfrac{1}{4}(v^j_{m,n}+v^j_{m+1,n}+2v^j_{m,n-1}+2v^j_{m+1,n-1});\\
&\text{Case D:}\quad \tilde{v}^j_{m,n}=\tfrac{1}{4}(2v^j_{m,n}+2v^j_{m+1,n}+v^j_{m,n-1}+v^j_{m+1,n-1});\\
&\text{Otherwise:}\quad \tilde{\mu}^j_{m,n}=\bar{\mu}^j_{m,n},\quad \tilde{v}^j_{m,n}=\bar{v}^j_{m,n}.
\end{aligned} \tag{22}$$

At velocity points adjacent and perpendicular to an open boundary, such as $(m+\frac{1}{2}, n)$ in Figure 2, case (B), the four point averages (8) can be used provided the values of μ or v on the open boundary itself are maintained at zero.

Equations (17)–(19) may be stepped explicitly backwards in time. If j is odd, equation (18) can be solved explicitly for $\mu^j_{m,n}$, $(m, n)\in S_p$ and then equation (19) explicitly for $v^j_{m,n}$, $(m, n)\in S_q$. If j is

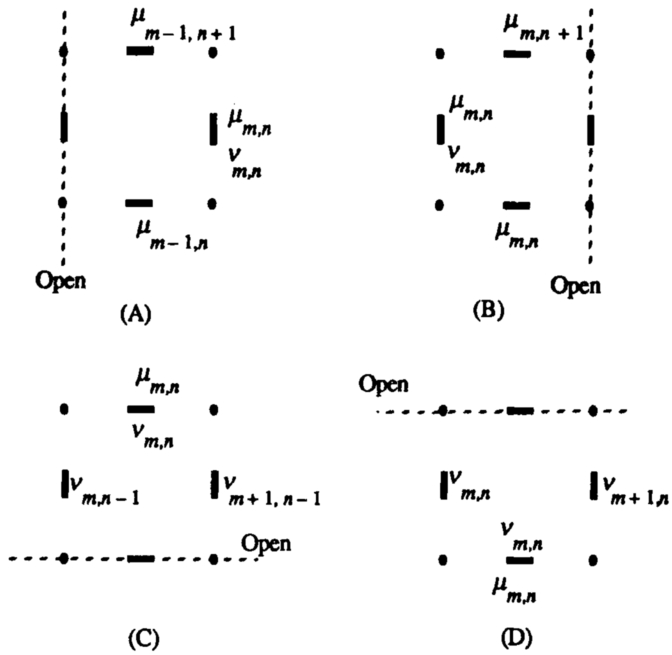

Figure 2. The four cases of μ and v points near open boundaries for which the averages $\tilde{\mu}_{m,n}$ and $\tilde{v}_{m,n}$ differ from the usual averages $\bar{\mu}_{m,n}$ and $\bar{v}_{m,n}$

even, the order in which (18) and (19) are solved is reversed. Then equation (17) can be solved explicitly for $\lambda^{j-1}_{m,n}$, $(m, n)\in S_\zeta$.

3.2. *Parameter determinations*

Having satisfied the adjoint system of equations, only the last two sums in equation (14) remain. We suppose that the quantities $k_{m,n}$ and $h_{m,n}$ are expressed, respectively, in terms of some reduced parameter sets $\{p_A: A=1, \ldots, N\}$ and $\{p_A: A=N+1, \ldots, 2N\}$ by expressions of the form

$$k_{m,n}=\sum_{A=1}^{N} M_{m,n,A}\, p_A, \qquad h_{m,n}=\sum_{A=1}^{N} M_{m,n,A}\, p_{N+A}. \tag{23}$$

The simplest form would be the piecewise linear approximations within triangular elements, with $\{p_A\}$ being the nodal values, and this is the form we have used in the numerical tests to be described below. Then we obtain

$$\delta F=\sum_{A=1}^{N}\{F_A\delta p_A+F_{N+A}\delta p_{N+A}\},$$

where

$$F_A=\sum_{(m,n)\in S_p} P_{m,n}\tfrac{1}{2}(M_{m,n,A}+M_{m+1,n,A})+\sum_{(m,n)\in S_q} Q_{m,n}\tfrac{1}{2}(M_{m,n,A}+M_{m,n+1,A}), \tag{24}$$

$$F_{N+A}=\sum_{(m,n)\in S_p} R_{m,n}\tfrac{1}{2}(M_{m,n,A}+M_{m+1,n,A})+\sum_{(m,n)\in S_q} S_{m,n}\tfrac{1}{2}(M_{m,n,A}+M_{m,n+1,A}). \tag{25}$$

These expressions provide the components of the gradient of F in the reduced parameter space.

Das and Lardner[24] have examined a number of minimization algorithms for parameter estimation in the case of unidirectional flow in a narrow channel. They have compared the conjugate-gradient method with or without Beale restarts, the secant method, the BFGS quasi-Newton method and simple direct iteration along the direction of steepest descent, and have found that the BFGS algorithm is, in most cases, about as fast as or faster than the other methods and has a much larger domain of convergence. Among these methods, therefore, we have here used only BFGS algorithm in the version contained in the subroutine CONMIN of Shanno and Phua.[26] We have also experimented with the truncated Newton algorithm of Nash.[27] Descriptions of these methods are given by Navon and Legler,[28] and Nash and Nocedal.[29]

In the problems examined, in which the dimension of the parameter space was relatively small (not more than 20), there was no significant difference between these two algorithms, either in the accuracy of the estimations or in the number of iterations (that is, gradient evaluations) needed for convergence.

4. NUMERICAL TESTS

A number of numerical examples have been used to test the effectiveness of the method described in Section 3. The model region consists of a rectangular gulf, open along one side. The dimensions of the gulf were taken as 15 grids in the x-direction by 14·5 grids in the y-direction with the grid size being 40 000 m. The open boundary points are $(m, 16)$, $2 \le m \le 16$. The boundary condition on the open boundary was taken as $\zeta^j_{m,16} = \sin(\omega j \Delta t)$, where $\omega = 2\pi/T$ and $T = 12$ h. The time step used was 360 s and the value of the Coriolis parameter was $f = 1{\cdot}22 \times 10^{-4}$.

In each case, the 'observed' solution was computed using the algorithm described in Section 2. During this computation, the program was allowed to run for a large number of steps (in practice 1440, or six days of real time) until the solution settled down to a periodic form, accurate to at least six digits. The values from the final period were used as observed values for the parameter estimation.

For the inverse computation, the start-up interval I was chosen shorter than that used to compute the observed solution, but was sufficient to provide a solution that is periodic in the last period to within errors of at most 0·02%. (In practice, $I = 840$ was found sufficient, and the window of observation was from step 841 to 960.) On the initial iteration, the parameters to be estimated were given certain 'guessed' values.

In each case, the data stations were given equal weights, which can be taken arbitrarily, and the value $K_d = \Delta t^{-1}$ was used in order to simplify equation (17).

The parameters were estimated using linear triangular elements with a total of four, five or nine nodes. The arrangement of the elements in each case is shown in Figure 3. In the first sets of results quoted below, the number of data stations was chosen rather arbitrarily as 20 for the four-node cases, 5 for the five-node cases and 24 for the nine-node cases. Subsequently, results of tests with smaller numbers of data stations will be given.

In the first series of tests, the parameters were given constant values, $h = 65$ m and $k = 0{\cdot}25 \times 10^{-5}$ m^{-2}. In each case, initial guesses of $h = 60$ m and $k = 0{\cdot}20 \times 10^{-5}$ m^{-2} were used, and a tolerance (the norm of the gradient of F at which the iterations stop) was set at 10^{-5}. The results obtained using the BFGS algorithm are as follows. Comparable results were obtained using Nash's truncated Newton algorithm.

With four nodes, the value of F was reduced from $0{\cdot}52 \times 10^{-1}$ to $0{\cdot}11 \times 10^{-7}$ in 51 iterations. The estimated values of the depth at the nodes were between 64·993 and 65·010, and the estimates of friction coefficient between $0{\cdot}24990 \times 10^{-5}$ and $0{\cdot}25006 \times 10^{-5}$.

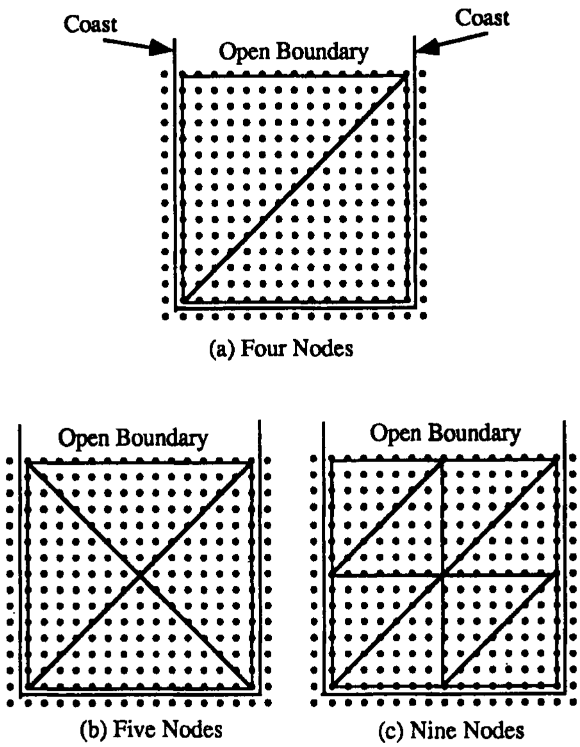

Figure 3. The finite element triangulations used for four, five and nine nodes. The points indicated by dots are the (m, n) grid points at which $\zeta_{m,n}$ is specified and parameter values assigned via equation (23)

With five nodes, F was reduced from $0{\cdot}19 \times 10^{-1}$ to $0{\cdot}77 \times 10^{-8}$ in 66 iterations. The estimated values of the depth at the nodes were between 64·890 and 65·019, and the estimates of friction coefficient between $0{\cdot}24952 \times 10^{-5}$ and $0{\cdot}25070 \times 10^{-5}$.

With nine nodes, F was reduced from $0{\cdot}39 \times 10^{-1}$ to $0{\cdot}18 \times 10^{-7}$ in 90 iterations. The estimated values of the depth at the nodes were between 64·862 and 65·112, and the estimates of friction coefficient between $0{\cdot}24903 \times 10^{-5}$ and $0{\cdot}25161 \times 10^{-5}$.

In the second series of tests, the parameters were given values that increased linearly with $m+n$ from smallest values of $h=33$ m and $k=0{\cdot}14 \times 10^{-5}\,\mathrm{m}^{-2}$ to largest values of $h=89$ m and $k=0{\cdot}42 \times 10^{-5}\,\mathrm{m}^{-2}$. In each case, initial guesses of constant values, $h=60$ m and $k=0.20 \times 10^{-5}\,\mathrm{m}^{-2}$, were used, and the tolerance 10^{-5} as before.

The results obtained using the BFGS algorithm are as follows. With four nodes, the value of F was reduced from 0·31 to $0{\cdot}10 \times 10^{-7}$ in 51 iterations. With five nodes, F was reduced from 0·13 to $0{\cdot}18 \times 10^{-8}$ in 70 iterations. With nine nodes, F was reduced from 0·22 to $0{\cdot}39 \times 10^{-7}$ in 118 iterations. The estimated and true values of the parameters at the nodes for the three cases are given in Table I. It can be seen that the algorithm converges to good estimates in each case, in spite of the initial guesses being very far out at some of the nodes. The errors are larger for nine nodes than for four or five, but are still reasonably small.

In similar parameter estimations for flow in a one-dimensional channel, Das and Lardner[24] found also that good values were obtained provided that the number of data stations was large enough. In their case, for both linear and quadratic bottom friction, joint estimates of depth and

Table I. Estimated and true parameters with four, five or nine nodes

Node		Friction coefficient ($\times 10^5$)		Depth	
m	*n*	Estimated	True	Estimated	True
2	2	0·14001	0·14	32·999	33
16	2	0·27998	0·28	61·000	61
16	16	0·42001	0·42	89·000	89
2	16	0·27999	0·28	61·000	61
2	2	0·14000	0·14	32·994	33
16	2	0·28000	0·28	60·993	61
9	9	0·28004	0·28	61·021	61
2	16	0·28005	0·28	61·000	61
16	16	0·42004	0·42	88·984	89
2	2	0·14039	0·14	32·992	33
9	9	0·27954	0·28	61·007	61
16	16	0·41949	0·42	88·994	89
9	2	0·21054	0·21	46·972	47
16	9	0·35190	0·35	74·963	75
16	2	0·27387	0·28	61·069	61
2	9	0·20991	0·21	47·019	47
9	16	0·35064	0·35	75·048	75
2	16	0·27963	0·28	60·956	61

friction coefficient were found to be good only provided the number of data stations was at least equal to the number of nodes. To ascertain whether a corresponding property holds for the present two-dimensional case, a series of tests was made with decreasing number of data points. The same parameter functions and starting guesses were used as in the last described tests.

Table II gives the results obtained for four nodes with four, three, two and one data point. It is seen that, in contrast to the one-dimensional case, there is very little increase in the mean error of the estimates as the number of data stations is reduced from four to two. It is only when the number of stations is reduced to one that the estimates become unreliable.

The corresponding results for five and nine nodes are summarized, respectively, in Tables III and IV. In these tables, only the mean errors in the estimates of k and h are given. It is seen that for five nodes, good estimates are obtained using three or more data stations, but with two or one, the errors become very large. For nine nodes, reliable estimates are obtained with six or more data stations, but not with five.

In the case of linear bottom friction with a single frequency solution, there are precisely two pieces of information contained in the values of surface elevations at each data station, namely the amplitude and phase. The result found by Das and Lardner,[24] that the total number of nodal parameter values to be estimated should not exceed twice the number of data stations, was therefore no surprise. The fact that these authors found a similar result for quadratic friction was somewhat surprising, and was ascribed by them to the relative weakness of the second harmonic generated by the non-linear friction term.

It is clear from Tables II–IV that, in the present two-dimensional problem, sufficient information is contained in the higher harmonics to raise the ratio, R, of the number of nodal parameter

Table II. Parameter estimations with four nodes and four, three, two and one data point. The upper half of the table contains estimates of $k \times 10^5$ and the lower half those of h

Node		True	$D=4$	$D=3$	$D=2$	$D=1$
2	2	0·14	0·14006	0·14013	0·13997	0·13796
16	2	0·28	0·27996	0·27994	0·28016	0·23432
16	16	0·42	0·42002	0·42004	0·42016	0·43178
2	16	0·28	0·27989	0·27979	0·27988	0·26896
Mean error			0·00006	0·00011	0·00012	0·01764
2	2	33	33·002	33·000	33·003	33·053
16	2	61	60·992	61·000	61·001	58·696
16	16	89	88·999	88·986	88·984	85·831
2	16	61	60·995	61·001	60·997	62·888
Mean error			0·004	0·004	0·006	1·856
No. of iterations			58	63	65	44
Initial value of F			$0·48 \times 10^{-1}$	$0·42 \times 10^{-1}$	$0·53 \times 10^{-1}$	$0·26 \times 10^{-1}$
Final value of F			$0·25 \times 10^{-8}$	$0·23 \times 10^{-8}$	$0·15 \times 10^{-8}$	$0·16 \times 10^{-6}$

Table III. Mean errors in estimates of the parameters ($k \times 10^5$ and h) with five nodes and five, four, three, two and one data point

	$D=5$	$D=4$	$D=3$	$D=2$	$D=1$
Mean error in k	0·00003	0·00016	0·00015	0·04913	0·04809
Mean error in h	0·012	0·014	0·017	4·921	8·153
No. of iterations	70	78	85	57	59
Initial value of F	$0·13 \times 10^{0}$	$0·72 \times 10^{-1}$	$0·11 \times 10^{0}$	$0·54 \times 10^{-1}$	$0·49 \times 10^{-1}$
Final value of F	$0·18 \times 10^{-8}$	$0·13 \times 10^{-8}$	$0·73 \times 10^{-9}$	$0·11 \times 10^{-5}$	$0·31 \times 10^{-7}$

Table IV. Mean errors in estimates of the parameters ($k \times 10^5$ and h) with nine nodes and decreasing number of data points

	$D=24$	$D=8$	$D=7$	$D=6$	$D=5$
Mean error in k	0·00122	0·00211	0·00399	0·00304	0·01784
Mean error in h	0·031	0·225	0·130	0·127	1·129
No. of iterations	118	112	101	127	132
Initial value of F	$0·22 \times 10^{0}$	$0·15 \times 10^{0}$	$0·13 \times 10^{0}$	$0·13 \times 10^{0}$	$0·13 \times 10^{0}$
Final value of F	$0·39 \times 10^{-7}$	$0·43 \times 10^{-7}$	$0·61 \times 10^{-7}$	$0·25 \times 10^{-7}$	$0·16 \times 10^{-6}$

values estimated to the number of data stations above the value two. For four nodes, R is four, and a ratio of four is also consistent with the results in Table III. But for nine nodes, R is less than four, so we must conclude that the second harmonic does not carry the full information of independent amplitude and phase at each data station, at least when the number of nodes is large.

If this argument is correct, we should expect that R would rise if the solution contains more than one fundamental frequency. We have, therefore, repeated some of the estimates with the open-boundary condition changed to $\zeta^j_{m,16} = \sin(\omega j \Delta t) + \sin(3\omega j \Delta t)$. Some of the critical results are given in Table V, where again only the mean errors in the parameter estimates are given.

Table V. Mean errors in estimates of the parameters ($k \times 10^5$ and h) with various numbers of nodes and data points for boundary condition containing two frequencies

	Four nodes	Five nodes		Nine nodes	
	$D=1$	$D=2$	$D=1$	$D=4$	$D=3$
Mean error in k	0·00052	0·00010	0·08005	0·00102	0·00971
Mean error in h	0·024	0·003	3·615	0·012	1·073
No. of iterations	82	82	201	105	110
Initial value of F	$0{\cdot}26 \times 10^{0}$	$0{\cdot}94 \times 10^{-1}$	$0{\cdot}89 \times 10^{-1}$	$0{\cdot}12 \times 10^{0}$	$0{\cdot}13 \times 10^{0}$
Final value of F	$0{\cdot}21 \times 10^{-8}$	$0{\cdot}46 \times 10^{-8}$	$0{\cdot}29 \times 10^{-5}$	$0{\cdot}98 \times 10^{-8}$	$0{\cdot}16 \times 10^{-6}$

For four nodes good estimates are now obtained with a single node, whereas with the monofrequency boundary input they were not (see Table II). For five nodes, good estimates are obtained now with two data stations, but not with one. For nine nodes good estimates are obtained with four data stations, but not with three.

With the chosen boundary condition, the quadratic friction generates two second harmonics and one independent combination frequency at the lowest order, giving a total of five frequencies in the solution. So, one might anticipate a possible ratio as high as $R = 10$. The actual ratios achieved are $R = 8$ for four nodes, $R = 5$ for five nodes and $R = 4{\cdot}5$ for nine nodes. This represents an improvement over the monofrequency case, but indicates that the harmonics are too weak to carry their full information.

5. SUMMARY AND DISCUSSION

A method has been described to estimate position-dependent parameters in a numerical tidal model by assimilating data from tide gauges. The method is based on an optimal control approach whereby a norm of the discrepancies between computed and measured values at the data stations is minimized, subject to the condition that the boundary value problem represented by the model equations is satisfied. The numerical algorithm we have used is a two-level leapfrog scheme, for which the adjoint scheme turns out to be a similar two-level leapfrog, but stepped backwards in time. We have tested two packages for the numerical optimization, the BFGS method contained in the CONMIN program of Shanno and Phua[26] and Nash's truncated Newton package.[27] These programs performed equally successfully, though it might be anticipated that for larger numbers of parameters than we have tested the Nash program would perform better.

The parameters estimated are the bottom friction coefficient and depth correction. Parameter functions are represented by finite element approximations; in the numerical tests we have made, piecewise linear approximations over triangular elements have been used.

In test problems on a model region, the method has turned out to provide very accurate estimates of the parameters, provided the number of data stations is sufficiently large. Some deterioration in accuracy is found as the number of estimated nodal parameter values is increased. Unlike the corresponding results found earlier for one-dimensional channel flow,[24] we have found that reliable estimates can be obtained with fewer data stations than finite element nodes. For boundary conditions containing a single frequency, the ratio of the number of nodal parameter values that can be accurately estimated to the number of data stations decreases from 4 for four nodes to 3 with nine nodes. For two independent frequencies in the boundary input, this ratio is 8 for four nodes, 5 for five nodes and 4·5 for nine nodes.

These results indicate that for the problem investigated, a significant amount of information is carried by the higher harmonics. However, the amount is less than what might be expected from the amplitude and phase for the fundamentals and all the harmonics generated at the lowest order. Nevertheless, the results are very encouraging for practical application, since real tidal data contain a very large number of constituents, and one might anticipate that accurate parameter estimates can be obtained from relatively few tidal data stations.

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